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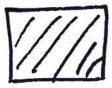
Last time... "Optimization Problem" (2D)

$$\min_R / \max \quad f = f(x,y)$$

$R =$  or 

Procedures:

Step 1: Locate all interior critical pts of f

Solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow (x,y) = \underset{P_1}{(x_1,y_1)}, \dots, \underset{P_k}{(x_k,y_k)}$

(collect those pts f is NOT diff.)

Step 2: Locate possible max/min on ∂R.

$\Rightarrow$ reduce to 1D-case. (may use polar coordinates)

$\Rightarrow$ 1D max/min. locate possible min/max pts, $q_1, \dots, q_\ell$.

Step 3: Compare the values of f at $p_1, \dots, p_k$ & $q_1, \dots, q_\ell$.

$$f(p_1), f(p_2), \dots, f(p_k), f(q_1), \dots, f(q_\ell)$$

find min/max.

Ultimate Goal: How do we do "Calculus" for

$$f(x_1, \dots, x_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) \,?$$

- Limit (concept of ∞)
- Derivative $\leftarrow$ "2D-derivative", $m = 1$.
- Integration

Q: How to talk about "differentiability" of

$$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$$

and "f'" ?

Goes back to 1D: $f = f(x)$, fix x_0.

(*)

f diff. at x_0 $\Leftrightarrow$ $\boxed{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists}}$ and $= f'(x_0)$

Naive approach to 2D: $f = f(x,y)$, fix $(x_0, y_0) = \vec{x}_0$, $\vec{x} = (x,y)$

f diff. at $\vec{x}_0$ $\Leftrightarrow$ $\lim_{\vec{x} \to \vec{x}_0} \frac{f(\vec{x}) - f(\vec{x}_0)}{\vec{x} - \vec{x}_0}$ exists.

vector (cannot divide by a vector)

Idea: Can we express (*) without "dividing" ?

Define a function:

$$\varepsilon(x) := f(x) - f(x_0) - f'(x_0)(x - x_0)$$

$\varepsilon(x) \to 0$
$x \to x_0$

$$\boxed{\lim_{x \to x_0} \frac{\varepsilon(x)}{x - x_0} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = 0} \quad (**)$$

(*) $\Leftrightarrow$ (**)

Def$^{\underline{n}}$: A function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0) = \vec{x}_0$

if $\exists$ some vector "$\nabla f(\vec{x}_0)$" $\in \mathbb{R}^2$ s.t.

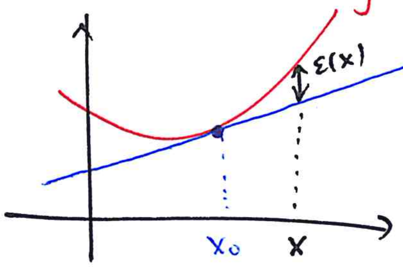
~~and~~ $\varepsilon(\vec{x}) := f(\vec{x}) - f(\vec{x}_0) - \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$

and $\lim_{\vec{x} \to \vec{x}_0} \frac{\varepsilon(\vec{x})}{\underbrace{\| \vec{x} - \vec{x}_0 \|}_{\text{number}}} = 0$

Q: How to understand it geometrically?

(1D):

$y = f(x)$



$y = f(x_0) + f'(x_0)(x - x_0)$

"f is diff. at x_0"

$\Updownarrow$

"The tangent line approximates the graph $y = f(x)$ up to order 1 at x_0".

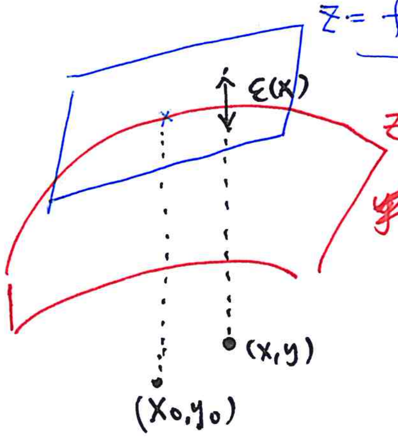
$\Updownarrow$ $\boxed{\lim_{x \to x_0} \frac{\varepsilon(x)}{x - x_0} = 0}$

"$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ near x_0"

(2D)

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$z = f(x, y)$



$\downarrow$ vector form.

Let $\vec{x} = (x, y)$, $\vec{x}_0 = (x_0, y_0)$.

$$z = f(\vec{x}_0) + \underbrace{(f_x(x_0, y_0), f_y(x_0, y_0))}_{\text{"}\nabla f(x_0,y_0)\text{"}} \cdot (\vec{x} - \vec{x}_0)$$

gradient vector.

Def: $f = f(x_1, \dots, x_n) : \mathbb{R}^n \to \mathbb{R}$

$$\nabla f(\vec{x}_0) := (f_{x_1}(\vec{x}_0), f_{x_2}(\vec{x}_0), \dots, f_{x_n}(\vec{x}_0)).$$

gradient vector of f at $\vec{x}_0$.

E.g. $f(x, y, z) = xy + z^2$. fix $\vec{x}_0 = (1, 1, 1)$

find $\nabla f(\vec{x}_0)$.

Sol: Compute 1st partial derivatives:

$$\left.\begin{aligned} f_x &= y \\ f_y &= x \\ f_z &= 2z \end{aligned}\right\} \xrightarrow{\text{at } \vec{x}_0 = (1,1,1)} \nabla f(\vec{x}_0) = (1, 1, 2)$$

same dimension as # of variables.

※

Example 1 : Show that $f(x,y) = x^2 + y^2$ is differentiable at $\underbrace{(0,0)}_{= \vec{x}_0}$.

Sol: "Proof from the def$^{\underline{n}}$".

Define our vector "$\nabla f(0,0)$" to be just the gradient vector:

$$\nabla f(0,0) = \left(f_x(0,0), f_y(0,0)\right) = (2x, 2y)\Big|_{(0,0)} = (0,0).$$

$$\varepsilon(\vec{x}) := f(\vec{x}) - \underbrace{f(0,0)}_{=0} - \underbrace{\nabla f(0,0)}_{=(0,0)} \cdot (\vec{x} - \vec{x}_0)$$

$$\Rightarrow \quad \varepsilon(\vec{x}) = f(\vec{x}) = x^2 + y^2.$$

Check: $\displaystyle \lim_{\vec{x} \to (0,0)} \frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{x}_0\|} = 0$

$$\frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{x}_0\|} = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2} \xrightarrow[\text{as } \vec{x} \to (0,0)]{\text{2D-limit}} 0$$

$\blacksquare$

Last time: Differentiability of $f = f(x,y)$ at (x_0, y_0).

$$f(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{L(x,y)\text{ : linear approximation}} + \mathcal{E}(x,y)$$

f is diff. at (x_0,y_0) $\Longleftrightarrow$ $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{\mathcal{E}(x,y)}{\|(x,y)-(x_0,y_0)\|} = 0$

Q: (1D): f is diff at x_0 $\Longleftrightarrow$ $\displaystyle\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. $= f'(x_0)$

(2D): f is diff at $\vec{x}_0$ $\overset{?}{\Longleftrightarrow}$ $\left[\displaystyle\lim_{\vec{x}\to\vec{x}_0} \frac{f(\vec{x})-f(\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|} \text{ exists}\right]$ not a good def.

[Eg. $f(x,y) = \cancel{\;\;}\; x+2y$.] [Q: when does this limit exist?]

Note: $\displaystyle\lim_{x\to x_0} \frac{f(x)-f(x_0)}{|x-x_0|}$ is not a good def$^{\underline{n}}$ for $f'(x_0)$.

Let $f(x) = x$, $x_0 = 0$.

$\displaystyle\lim_{x\to 0} \frac{f(x)-f(x_0)}{|x-x_0|} = \lim_{x\to 0}\frac{x}{|x|}$ does not exist.

Def$^{\underline{n}}$: A function $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}_0$

if $\exists$ $L(\vec{x}) := f(\vec{x}_0) + \vec{v}\cdot(\vec{x}-\vec{x}_0)$ linear approximation.

s.t. if $\varepsilon(\vec{x}) := f(\vec{x}) - L(\vec{x})$ error

then $\displaystyle\lim_{\vec{x}\to\vec{x}_0} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{x}_0\|} = 0$

Q: what is $\vec{v}$?

A: $\vec{v} = \nabla f(\vec{x}_0)$

$(n=2)$

Theorem: If $f(x,y)$ is diff. at (x_0,y_0)

then (1) $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist.

(2) and $\vec{V} = (f_x, f_y)\Big|_{(x_0,y_0)} =: \nabla f(x_0,y_0)$

$\uparrow$ in def^{n} of diff.

$$\boxed{f \text{ is diff. at } \vec{x}_0 \Rightarrow f_{x_1},\dots,f_{x_n} \text{ exist at } \vec{x}_0}$$

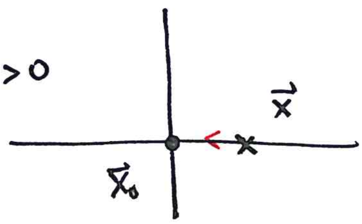
Remark. " $\Leftarrow$ " is false.

e.g. $f(x,y) = \begin{cases} 0 & \text{if } xy=0 \\ 1 & \text{if } xy\neq 0 \end{cases}$

is not even continuous at $(0,0)$.

Proof of Thm:

Let $\vec{x}_0 = (x_0,y_0)$, $\boxed{\vec{x} = (x_0+h, y_0)}$, $h>0$



As $h\to 0$, $\vec{x}\to\vec{x}_0$.

From of def^{n}, $\vec{V} = (V_1, V_2)$ Goal: $V_1 = f_x$, $V_2 = f_y$ at $\vec{x}_0$

$L(\vec{x}) := f(\vec{x}_0) + \vec{V}\cdot(\vec{x}-\vec{x}_0)$

$$\frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{x}_0\|} = \frac{f(\vec{x})-L(\vec{x})}{\|\vec{x}-\vec{x}_0\|}$$

as $\vec{x}\to\vec{x}_0$, $h\to 0$ $\to 0$, f is diff at $\vec{x}_0$.

$$= \frac{f(\vec{x})-f(\vec{x}_0)-\vec{V}\cdot(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|}$$

In our case,
$\vec{x}-\vec{x}_0 = (h,0)$
$\|\vec{x}-\vec{x}_0\| = h > 0$

$$= \frac{f(x_0+h,y_0)-f(x_0,y_0)-hV_1}{h}$$

$$= \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} - V_1$$

$\downarrow h\to 0$ $\qquad\qquad$ $\downarrow h\to 0$

$f_x(x_0,y_0)$ $\qquad\qquad$ $-V_1$ $\quad\Rightarrow\quad$ $V_1 = f_x(x_0,y_0)$. $\blacksquare$

Similarly, you get

$V_2 = f_y(x_0,y_0)$.

Thm: f is diff. at $\vec{x}_0 \Rightarrow f$ is cts at $\vec{x}_0$.

Pf: Goal: f is cts at $\vec{x}_0$

$$\Longleftrightarrow \boxed{\underbrace{\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})}_{\text{2D-limit}} = f(\vec{x}_0)} \quad (*)$$

exists & equal to $f(\vec{x}_0)$

f is diff. at x_0 $\overset{\text{def}^{\text{n}}}{\Longleftrightarrow}$ $$\lim_{\vec{x}\to\vec{x}_0} \frac{f(\vec{x}) - f(\vec{x}_0) - \nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|} = 0$$

$$f(\vec{x}) - f(\vec{x}_0) = \left[\frac{f(\vec{x}) - f(\vec{x}_0) - \nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|}\right] \|\vec{x}-\vec{x}_0\|$$
$$+ \underbrace{\nabla f(\vec{x}_0)}\cdot(\vec{x}-\vec{x}_0)$$

$$\xrightarrow{\text{as } \vec{x}\to\vec{x}_0} [0]\cdot[0] + \underbrace{\nabla f(\vec{x}_0)\cdot\vec{0}}_{=0} = 0$$

$$\Rightarrow \quad \lim_{\vec{x}\to\vec{x}_0} \left(f(\vec{x}) - f(\vec{x}_0)\right) = 0 \quad \Longleftrightarrow \quad (*)$$

∎

In summary,

$\boxed{?} \Rightarrow \boxed{f \text{ is diff. at } \vec{x}_0}$

$\nearrow$ $\boxed{f_x, f_y \text{ exist at } \vec{x}_0}$

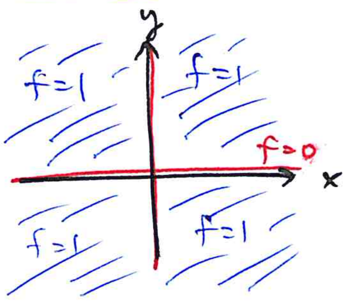
$\not\Uparrow \quad \not\Downarrow \quad$ Q?

$\Rightarrow$ $\boxed{f \text{ is cts at } \vec{x}_0}$

$$f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0 \end{cases}$$

$f_x(0,0) = f_y(0,0) = 0$.

f is not cts at $(0,0)$



Yes, $\boxed{?} = \boxed{f \text{ is } C^1 \text{ at } \vec{x}_0}$

[<u>Def$^{\underline{n}}$</u>: f is $\boxed{C^k}$ at $\vec{x}_0$

if all the k-th order partial derivatives exist (at $\vec{x}_0$)

and are still continuous. at $\vec{x}_0$.]

<u>E.g.</u> $f(x,y)$ is C^1 at $\vec{x}_0$

$\Leftrightarrow$ f_x and f_y <u>exists</u> ① and they are <u>cts</u> ② at $\vec{x}_0$.

<u>Picture</u>: <u>$n=2$</u>:

$$f \text{ is diff at } \vec{x}_0 \quad \Rightarrow \quad f_x, f_y \text{ exist at } \vec{x}_0$$
$$\not\Leftarrow$$

but it's true

if f_x, f_y cts at $\vec{x}_0$

<u>E.g.</u> $f(x,y) = e^x \sin xy$.

<u>Q</u>: Is f diff. at $(0,0)$?

$$\left.\begin{aligned} f_x &= e^x \sin xy + y e^x \cos xy \\ f_y &= x e^x \cos xy \end{aligned}\right\} \text{ cts everywhere.}$$

$\Rightarrow$ f is C^1 everywhere

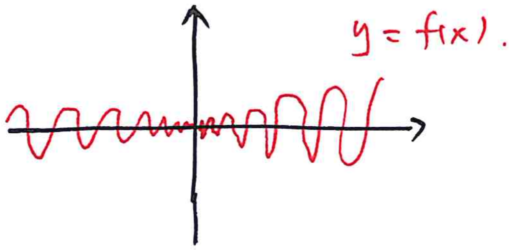
$\Rightarrow$ f is diff everywhere.

[<u>Q</u>: f is diff. at $\vec{x}_0$ $\not\Rightarrow$ f is C^1 at $\vec{x}_0$] <u>No!</u>

$\Leftarrow$ true.

Counterexample (1D)

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$



Claim: (1) f is differentiable everywhere, $f'(x)$ exists for all x.

(2) $f'(x)$ is not cts at $x=0$. (ie $f \notin C^1$ at $x=0$)

Pf: f is clearly diff. at $x \neq 0$.

$$f'(x) = \left(x^2 \sin \frac{1}{x}\right)' = 2x \sin \frac{1}{x} + x^2 \left(-\frac{1}{x^2}\right) \cos \frac{1}{x}$$

$$= 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

At $x = 0$,

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0 \quad \text{(sandwich thm.)}.$$

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

this is cts at $x = 0$.

but $\lim_{x \to 0} \left(\underset{\downarrow \, 0}{2x \sin \frac{1}{x}} - \underbrace{\cos \frac{1}{x}}_{\text{does not have limit}} \right)$ does not exist.

[Ex: Find a function f which is k-times differentiable but $f^{(k)}$ is not cts.]
(1D)

Example: Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Prove (i) f is cts at $(0,0)$.

(ii) $f_x(0,0)$, $f_y(0,0)$ exist and $= 0$.

(iii) **Q**: Is f diff. at $(0,0)$?.

Sol$^{\underline{n}}$ (i) **Goal**: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

use polar coord: $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r} = \lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{r} = 0.$$

\#.

(ii)

$$f_x(0,0) = \lim_{h\to 0} \frac{\overset{0}{\overset{\shortparallel}{f(h,0)}} - \overset{0}{\overset{\shortparallel}{f(0,0)}}}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{\overset{0}{\overset{\shortparallel}{f(0,h)}} - \overset{0}{\overset{\shortparallel}{f(0,0)}}}{h} = 0$$

(iii) **Picture**:

$$\boxed{f \in C^1 \text{ at } \vec{x}_0} \Rightarrow \boxed{f \text{ is diff. at } \vec{x}_0} \begin{array}{l} \nearrow \boxed{f_x, f_y \text{ exist at } \vec{x}_0} \\ \searrow \boxed{f \text{ is cts at } \vec{x}_0} \end{array}$$

For this example,

$\overset{?}{\boxed{\text{false}}}$ $\qquad$ $\overset{?}{\boxed{\text{false}}}$ $\qquad$ $\boxed{\text{true}}$ $\qquad$ $\boxed{\text{true}}$

↑ use def$^{\underline{n}}$

"worst-case scenario"

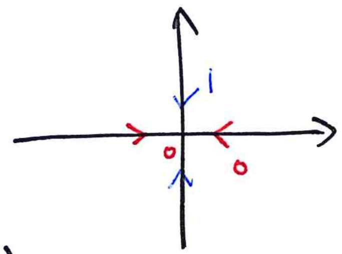
Check: $f \in C^1$ at $(0,0)$.

$$f_x(x,y) = \begin{cases} \dfrac{y^3}{(x^2+y^2)^{3/2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\left[\text{Calculation: } \frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \frac{\sqrt{x^2+y^2}\,(y) - xy\left(\cancel{\tfrac{1}{2}} \frac{1}{\sqrt{x^2+y^2}} \cdot \cancel{2}x\right)}{x^2+y^2} = \frac{(\cancel{x^2}+y^2)y - \cancel{x^2y}}{(x^2+y^2)^{3/2}} \right]$$

Along $y=0$,

$$f_x(x,0) = 0 \xrightarrow{\text{as } x\to 0} 0 = f_x(0,0)$$



Along $x=0$,

$$f_x(0,y) = \frac{y^3}{y^3} = 1 \xrightarrow{\text{as } y\to 0} 1 \neq f_x(0,0)$$

$\Rightarrow \lim_{(x,y)\to(0,0)} f_x(x,y)$ does <u>NOT</u> exist.

$\Rightarrow f_x$ is <u>NOT</u> cts at $(0,0)$.

From def$^{\underline{n}}$, $L(x,y) = \underbrace{f(0,0)}_{=0} + \underbrace{f_x(0,0)}_{=0}(x-0) + \underbrace{f_y(0,0)}_{=0}(y-0)$

$\Rightarrow L(x,y) = 0$.

$$\varepsilon(x,y) := f(x,y) - L(x,y) = f(x,y)$$

$$\frac{\varepsilon(x,y)}{\sqrt{x^2+y^2}} = \frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{xy}{x^2+y^2} \xrightarrow[\text{as } (x,y)\to(0,0)]{?} 0$$

f <u>NOT</u> diff. at $(0,0)$.

along $y=kx$, $\dfrac{kx^2}{(1+k^2)x^2} = \dfrac{k}{1+k^2} \Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ <u>NOT</u> exist.

depends on k

Q: what about differentiability of

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad \underline{m>1}.$$

$$f = (f_1, \dots, f_m)$$

function $\mathbb{R}^n \to \mathbb{R}$

Fact: f is diff. $\iff$ all $f_1, \dots, f_m$ are diff.

$\underline{m=1}$: $$Df = \nabla f = (f_{x_1}, \dots, f_{x_n}) \qquad 1\times n \text{ matrix}$$

$\underline{m>1}$: $$\text{``}Df\text{''} = \begin{pmatrix} (f_1)_{x_1} & \cdots & (f_1)_{x_n} \\ \vdots & & \vdots \\ (f_m)_{x_1} & \cdots & (f_m)_{x_n} \end{pmatrix}. \qquad m\times n \text{ matrix}.$$

E.g.: $f(x,y) = (x+y,\ x^2-y^2)$.

$$Df = \begin{pmatrix} \frac{\partial}{\partial x}(x+y) & \frac{\partial}{\partial y}(x+y) \\ \frac{\partial}{\partial x}(x^2-y^2) & \frac{\partial}{\partial y}(x^2-y^2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2x & -2y \end{pmatrix}$$